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A fractional Dirac equation and its solution

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Abstract

This paper presents a fractional Dirac equation and its solution. The fractional Dirac equation may be obtained using a fractional variational principle and a fractional Klein–Gordon equation; both methods are considered here. We extend the variational formulations for fractional discrete systems to fractional field systems defined in terms of Caputo derivatives. By applying the variational principle to a fractional action S, we obtain the fractional Euler–Lagrange equations of motion. We present a Lagrangian and a Hamiltonian for the fractional Dirac equation of order α. We also use a fractional Klein–Gordon equation to obtain the fractional Dirac equation which is the same as that obtained using the fractional variational principle. Eigensolutions of this equation are presented which follow the same approach as that for the solution of the standard Dirac equation. We also provide expressions for the path integral quantization for the fractional Dirac field which, in the limit $\alpha \to 1$, approaches to the path integral for the regular Dirac field. It is hoped that the fractional Dirac equation and the path integral quantization of the fractional field will allow further development of fractional relativistic quantum mechanics.

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1. Introduction

Fractional calculus is an emerging field and during the last few decades some important contributions were developed in science, engineering, applied mathematics, economics and biomechanics. In recent years, considerable progress has been made in the area of fractional derivatives and, in more general, in the area of fractional calculus. A complete discussion of

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this subject will take us too far from our main goal here [1–6]. Particularly, the fractional and anomalous dynamics has experienced an upswing during the last few years and has been forged into a mature framework in the theory of stochastic processes [7]. The physical and geometrical meaning of the fractional derivatives has been investigated by several authors [8, 9]. The physical interpretation of the Stieltjes integral, namely the Stieltjes integral can be interpreted as the real distance passed by a moving object, for which we have recorded correct values of speed and the incorrect values of time. Within this physical interpretation the Riemann–Liouville and Caputo have the same interpretation and therefore can be used successfully, for example,within fractional variational principles [10–17]. We mention that this interpretation allows us to use fractional derivatives inside of the fractional action. On the other hand, by making use of examples from viscoelasticity, it was shown that it is possible to attribute physical meaning to initial conditions within Riemann–Liouville fractional derivatives [18]. The fractional operators are particular cases of non-local operators and during the last decade several interesting research was done in this direction [19–25].

The Dirac equation is a relativistic quantum mechanical wave equation which provides a description of elementary spin-$\frac{1}{2}$ particles, such as electrons, consistent with both the principles of quantum mechanics and special theory of relativity [26]. One of the approaches of obtaining the Dirac equation is to take the square root of the operators appearing in the Klein–Gordon equation. The algebra of the coefficients in the proposed square root suggests that these coefficients should be 4 by 4 matrices. One of the major successes of this equation is that it predicted the existence of a positron (an antiparticle of an electron) before it was discovered [26].

Following the above progress, one can naturally ask: 'Can we take other roots of the Dirac and the Klein–Gordon equations to describe new systems'. This is exactly what was done in [27] and [28]. Ketov and Prager [27] examine the square root of the Dirac equation, and Raspini [28] proposed a fractional Dirac equation (FDE) of order $2/3$ which is essentially a cube root of the Klein–Gordon equation ('What is a fractional Dirac equation?' will be discussed shortly). The algebra of the coefficients in the Raspini's formulation suggests that these coefficients should be 9 by 9 matrices. Raspini [28] also provides simple solutions and the spinorial properties of the proposed FDE.

Raspini's formulation could be extended to derive FDEs of order m/n, where m and n, $m \leqslant n$, are two arbitrary integers. However, the approach suggests that for large n, the size of the matrices arising in the formulation would also be large. Furthermore, in Raspini's formulation, what would be the root of arbitrary order of the Klein–Gordon equation is not clear. Therefore, using Raspini's formulation, it may not be possible to obtain the Dirac's equations containing spacetime derivatives of arbitrary order. In addition, the Raspini's formulation does not provide Lagrangians and Hamiltonian for the Dirac equation of order $2/3$. Accordingly, it has not been extended in the direction that requires Lagrangians and Hamiltonian of a system.

The first problem could be avoided by considering a fractional Klein–Gordon equation as the starting point for the FDEs. This approach to derive an FDE will be considered later in this paper. As discussed later, it would also reduce the size of the matrices arising in the formulation. To address the second problem, one could develop fractional Lagrangians and Hamiltonians and use a fractional variational calculus to derive the fractional Dirac equation. This would be considered in the next three sections of this paper.

At this stage, one may ask, 'What are fractional Dirac and Klein–Gordon equations?' A *fractional Dirac (Klein–Gordon) equation* could be thought of as a Dirac (Klein–Gordon) equation in which regular space and time derivatives have been replaced by fractional space and time derivatives.

One may also ask, 'What is the fractional variational calculus?' To address this, we first define a variational problem. A *variational problem* is a problem that requires finding the extremum of a functional which may be subjected to algebraic and/or dynamic constraints. If either the functional and/or the algebraic/dynamic constraint(s) contain at least one fractional derivative term, then the problem is called a *fractional variational problem*. The *fractional variational calculus* is an extension of the ordinary variational calculus that deals with finding the solution of a fractional variational problem [10–17]. As pointed out above, we will define fractional Lagrangians and Hamiltonians, and use fractional variational calculus to derive the fractional Dirac equation and to address the issue of the quantization of the fractional Dirac's field.

The plan of this paper is as follows: In section 2, we present the Euler–Lagrange equations for a fractional field. Section 3 presents the Hamiltonian formulations of fields for three different cases discussed later. In section 4, we define a fractional Lagrangian density function and use the theories developed in sections 2 and 3 to derive the fractional Dirac equation and the fractional Hamiltonian equations for a fractional Dirac field. In section 5, we derive the fractional Dirac equation from a fractional Klein–Gordon equation, and show that these equations are the same as those derived in section 4. This section also presents the eigensolutions of the fractional Dirac equation. In section 6 we define path integral quantization of fractional Dirac's field with the hope that it will allow further development of the fractional quantum mechanics. Finally, section 7 presents conclusions.

At this stage, it should be pointed out that several definitions have been proposed of a fractional derivative; among those the Riemann–Liouville and Caputo fractional derivatives are the most popular. The differential equations defined in terms of Riemann–Liouville derivatives require fractional initial conditions whereas the differential equations defined in terms of Caputo derivatives require regular boundary conditions. For this reason, the Caputo fractional derivatives are popular among scientists and engineers. Accordingly, we shall also develop our formulations in terms of Caputo fractional derivatives. However, most of the approach will also be applicable to problems defined using other derivatives.

2. Lagrangian formulation of field systems with Caputo fractional derivatives

Consider a function f depending on n variables, $x_1, \ldots, x_n$ defined over the domain $\Omega = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Following the convention used in physics, we defined the left and the right partial Riemann–Liouville and Caputo fractional derivatives of order α_k, $0 < \alpha_k < 1$ with respect to x_k as

$$\left({}_{+}\partial_k^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha_k)} \partial x_k \int_{a_k}^{x_k} \frac{f(x_1, \ldots, x_{k-1}, u, x_{k+1}, \ldots, x_n)}{(x_k - u)^{\alpha_k}} \, \mathrm{d}u, \tag{1}$$

$$\left({}_{-}\partial_k^{\alpha} f\right)(x) = \frac{-1}{\Gamma(1-\alpha_k)} \partial x_k \int_{x_k}^{b_k} \frac{f(x_1, \ldots, x_{k-1}, u, x_{k+1}, \ldots, x_n)}{(u - x_k)^{\alpha_k}} \, \mathrm{d}u, \tag{2}$$

$$\left({}_{+}^{C}\partial_k^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha_k)} \int_{a_k}^{x_k} \frac{\partial_u f(x_1, \ldots, x_{k-1}, u, x_{k+1}, \ldots, x_n)}{(x_k - u)_k^{\alpha}} \, \mathrm{d}u \tag{3}$$

and

$$\left({}_{-}^{C}\partial_k^{\alpha} f\right)(x) = \frac{-1}{\Gamma(1-\alpha_k)} \int_{x_k}^{b_k} \frac{\partial_u f(x_1, \ldots, x_{k-1}, u, x_{k+1}, \ldots, x_n)}{(u - x_k)^{\alpha_k}} \, \mathrm{d}u, \tag{4}$$

where $\partial x_k g$ is the partial derivatives of g with respect to the variable x_k. Here, in ${}_{+}\partial_k^{\alpha}$, ${}_{-}\partial_k^{\alpha}$, ${}_{+}^{C}\partial_k^{\alpha}$ and ${}_{-}^{C}\partial_k^{\alpha}$, the meaning of various subscripts and superscripts need to be made clear.

The subscript k and the superscript α indicate that the derivative is taken with respect to the variable x_k and it is of order α_k (note that we write only α for α_k, and the subscript k to ∂ also represents the subscript to α), the subscripts + and − prior to the symbol ∂ represent the left and the right fractional derivatives, respectively, and accordingly the limits of integrations are taken as $[a_k, x_k]$ and $[x_k, b_k]$. Further, no superscript and the superscript C prior to the symbol ∂ represent the Riemann–Liouville fractional derivative and the Caputo fractional derivative, respectively. Superscript α is necessary here as a reminder that the operator ∂^α represents a fractional derivative. When α is equal to 1, the superscript α can be neglected. Although, our aim in this section is to present the action principle for systems defined in terms of Caputo fractional derivatives, the Riemann–Liouville fractional derivatives are also defined here because they naturally arise in the formulation.

To develop the action principle for field systems described in terms of fractional derivatives, define a functional $S(\phi)$ as

$$S(\phi) = \int \mathcal{L}\big(\phi(x_k), \big({}^{C}_{+}\partial^{\alpha}_{k}\big)\phi(x_k), \big({}^{C}_{-}\partial^{\beta}_{k}\big)\phi(x_k), x_k\big)\,(\mathrm{d}x_k), \tag{5}$$

where $\mathcal{L}\big(\phi(x_k), \big({}^{C}_{+}\partial^{\alpha}_{k}\big)\phi(x_k), \big({}^{C}_{-}\partial^{\beta}_{k}\big)\phi(x_k), x_k\big)$ is a Lagrangian density function. Here we have used Goldstein's [29] notation. Accordingly, x_k represents n variables x_1 to x_n, $\phi(x_k) \equiv \phi(x_1, \ldots, x_1)$, $\mathcal{L}\big(*, {}^{C}_{+}\partial^{\alpha}_{k}, *, *\big) \equiv \mathcal{L}\big(*, {}^{C}_{+}\partial^{\alpha}_{1}, \ldots, {}^{C}_{+}\partial^{\alpha}_{n}, *, *\big)$, $(\mathrm{d}x_k) \equiv \mathrm{d}x_1 \cdots \mathrm{d}x_n$, and the integration is taken over the entire domain Ω. Other terms are defined accordingly.

To find the necessary condition for extremum of the action functional defined above, consider a one-parameter family of possible functions $\phi(x_k; \epsilon)$ as follows:

$$\phi(x_k; \epsilon) = \phi(x_k; 0) + \epsilon\eta(x_k), \tag{6}$$

where $\phi(x_k; 0)$ is the correct function which satisfies the Hamilton's principle for the fractional system, $\eta(x_k)$ is a well-behaved function that vanishes at the endpoints and ϵ is an arbitrary parameter. Note that $S[\phi(x_k; \epsilon)]$ is extremum at $\epsilon = 0$. Substituting equation (6) into equation (5), differentiating the resulting expression with respect to ϵ, and then setting ϵ to 0, we obtain

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \int \left[\frac{\partial\mathcal{L}}{\partial\phi}\eta + \sum_{k=1}^{n} \frac{\partial\mathcal{L}}{\partial\big({}^{C}_{+}\partial^{\alpha}_{k}\phi\big)}\big({}^{C}_{+}\partial^{\alpha}_{k}\eta\big) + \sum_{k=1}^{n} \frac{\partial\mathcal{L}}{\partial\big({}^{C}_{-}\partial^{\beta}_{k}\phi\big)}\big({}^{C}_{-}\partial^{\beta}_{k}\eta\big)\right](\mathrm{d}x_k) = 0. \tag{7}$$

Finally, using the formula for integration by part [11], the fact that $\eta(x_k)$ is zero at the boundary, and a lemma from the calculus of variations, we obtain

$$\frac{\partial\mathcal{L}}{\partial\phi} + \sum_{k=1}^{n} {}_{-}\partial^{\alpha}_{k} \frac{\partial\mathcal{L}}{\partial\big({}^{C}_{+}\partial^{\alpha}_{k}\phi\big)} + \sum_{k=1}^{n} {}_{+}\partial^{\beta}_{k} \frac{\partial\mathcal{L}}{\partial\big({}^{C}_{-}\partial^{\beta}_{k}\phi\big)} = 0. \tag{8}$$

Equation (8) is the Euler–Lagrange equation for the fractional field system. For $\alpha_k, \beta_k \to 1$, equation (8) gives the usual Euler–Lagrange equations for classical fields.

Here we would like to address the following. (1) We have formulated the problem in terms of Caputo derivatives. The same approach can be used to find the Euler–Lagrange equations for functionals defined in terms of Riemann–Liouville or mixed (Caputo and Riemann–Liouville) derivatives. (2) The presence of right/left Caputo/Riemann–Liouville derivatives leads to left/right Riemann–Liouville/Caputo derivatives in the Euler–Lagrange equations. (3) In the formulation so far, ϕ was treated as a scalar function. The above equations are equally valid when ϕ is a vector function.

We now present Hamiltonian formulations for fractional field systems.

3. Hamiltonian formulation of fields within Caputo derivatives

Like discrete systems and integer-order fields, it is possible to develop Hamiltonian formulations for fields defined in terms of fractional derivatives. From mathematical point of view, a Hamiltonian formulation is nothing but a set of Legendre transformations which replaces the generalized velocities (or the generalized velocity densities, GVDs) from the Lagrangian (or the Lagrangian density) with generalized momenta (or momentum densities)[5]. The transformed Lagrangian (or the Lagrangian density) obtained through this process is called the Hamiltonian (or the Hamiltonian density). Since our discussion is limited to fields, we shall consider only field variables. The change of variables proposed above could be for the complete set or a partial set of generalized velocity densities. Standard Legendre approach cannot be used for such changes if the Lagrangian density function is a linear function of the generalized velocity densities. However, for this last case, using a Lagrange multiplier technique, it is still possible to develop the Hamiltonian density functions which give correct equations of motion.

In this section, we develop Hamiltonian formulations for the three cases (namely, the change of partial and complete set of GVDs and the change of GVDs when the Lagrangian density function is a linear function of the GVDs) for fractional fields. Although we develop the formulations for fields defined in terms of Caputo fractional derivatives, the approach is applicable to fields defined in terms of other fractional derivatives. In all cases considered here, we define the fractional canonical momentum densities (FCMDs) π_{α_k} and π_{β_k} as

$$\pi_{\alpha_k} = \frac{\partial \mathcal{L}}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)}, \qquad \pi_{\beta_k} = \frac{\partial \mathcal{L}}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)}. \tag{9}$$

Thus, in terms of π_{α_k} and π_{β_k}, the Euler–Lagrange equation is given as

$$\frac{\partial \mathcal{L}}{\partial \phi} + \sum_{k=1}^{n} {}_{-}\partial_k^{\alpha}\pi_{\alpha_k} + \sum_{k=1}^{n} {}_{+}\partial_k^{\beta}\pi_{\beta_k} = 0. \tag{10}$$

We now give the Hamiltonian formulations for the three cases.

Case 1. Change of a partial set of GVDs. Let us assume that we want to replace GVDs ${}_{+}^{C}\partial_k^{\alpha}\phi$ with the FCMD π_{α_k} $(k = 1, \ldots, n_l < n)$ and the GVDs ${}_{-}^{C}\partial_k^{\beta}\phi$ with the FCMDs π_{β_k} $(k = 1, \ldots, n_r < n)$. Before we proceed further, it should be pointed out that ${}_{+}^{C}\partial_k^{\alpha}\phi$ and ${}_{-}^{C}\partial_k^{\beta}\phi$ to be replaced with π_{α_k} and π_{β_k}, respectively, need not be in sequence. This assumption was made for simplicity. For this case, with define the Hamiltonian density function as

$$\mathcal{H} = \sum_{k=1}^{n_l} \pi_{\alpha_k}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right) + \sum_{k=1}^{n_l} \pi_{\beta_k}\left({}_{-}^{C}\partial_k^{\beta}\phi\right) - \mathcal{L}, \tag{11}$$

where $\mathcal{H}$ is now a function of ϕ; π_{α_k} $(k = 1, \ldots, n_l)$; ${}_{+}^{C}\partial_k^{\alpha}\phi$ $(k = n_l + 1, \ldots, n)$; π_{β_k} $(k = 1, \ldots, n_r)$; ${}_{-}^{C}\partial_k^{\alpha}\phi$ $(k = n_r + 1, \ldots, n)$; and x_k, $k = 1, \ldots, n$. The total differential of equation (11) is given as

$$\begin{aligned} \mathrm{d}\mathcal{H} = {} & \sum_{k=1}^{n_l}\left(\mathrm{d}\pi_{\alpha_k}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right) + \pi_{\alpha_k}\,\mathrm{d}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)\right) + \sum_{k=1}^{n_r}\left(\mathrm{d}\pi_{\beta_k}\left({}_{-}^{C}\partial_k^{\beta}\phi\right) + \pi_{\beta_k}\,\mathrm{d}\left({}_{-}^{C}\partial_k^{\beta}\phi\right)\right) \\ & - \sum_{k=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial x_k}\,\mathrm{d}x_k + \frac{\partial \mathcal{L}}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)}\,\mathrm{d}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right) + \frac{\partial \mathcal{L}}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)}\,\mathrm{d}\left({}_{-}^{C}\partial_k^{\beta}\phi\right)\right) - \frac{\partial \mathcal{L}}{\partial \phi}\,\mathrm{d}\phi. \end{aligned} \tag{12}$$

[5] The terms generalized velocity, generalized velocity density, canonical momentum density, etc are used in extended sense.

Using equations (9) and (10), equation (12) reduces to

$$\mathrm{d}\mathcal{H} = \sum_{k=1}^{n_l} \mathrm{d}\pi_{\alpha_k} \left({}_{+}^{C}\partial_k^{\alpha}\phi\right) + \sum_{k=1}^{n_r} \mathrm{d}\pi_{\beta_k} \left({}_{-}^{C}\partial_k^{\beta}\phi\right) - \sum_{k=1}^{n} \frac{\partial\mathcal{L}}{\partial x_k}\,\mathrm{d}x_k$$
$$- \sum_{k=n_1+1}^{n} \frac{\partial\mathcal{L}}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)}\,\mathrm{d}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right) - \sum_{k=n_r+1}^{n} \frac{\partial\mathcal{L}}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)}\,\mathrm{d}\left({}_{-}^{C}\partial_k^{\beta}\phi\right) - \frac{\partial\mathcal{L}}{\partial\phi}\,\mathrm{d}\phi. \tag{13}$$

Finally, after some manipulations, we obtain

$$\begin{aligned} \frac{\partial\mathcal{H}}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)} &= -\frac{\partial\mathcal{L}}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)}, \qquad k = n_l+1,\ldots,n, \\ \frac{\partial\mathcal{H}}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)} &= -\frac{\partial\mathcal{L}}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)}, \qquad k = n_r+1,\ldots,n \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\frac{\partial\mathcal{H}}{\partial\pi_{\alpha_k}} = \left({}_{+}^{C}\partial_k^{\alpha}\phi\right), \qquad k = 1,\ldots,n_l; \qquad \frac{\partial\mathcal{H}}{\partial\pi_{\beta_k}} = \left({}_{-}^{C}\partial_k^{\beta}\phi\right), \qquad k = 1,\ldots,n_r, \\ &\frac{\partial\mathcal{H}}{\partial x_k} = -\frac{\partial\mathcal{L}}{\partial x_k}, \qquad k = 1,\ldots,n, \\ &\frac{\partial\mathcal{H}}{\partial\phi} = \sum_{k=1}^{n_l} {}_{-}\partial_k^{\alpha}\pi_{\alpha_k} + \sum_{k=1}^{n_r} {}_{+}\partial_k^{\beta}\pi_{\beta_k} - \sum_{k=n_l+1}^{n} {}_{-}\partial_k^{\alpha}\frac{\partial\mathcal{H}}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)} - \sum_{k=n_r+1}^{n} {}_{+}\partial_k^{\beta}\frac{\partial\mathcal{H}}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)}. \end{aligned} \tag{15}$$

Equations (15) are called the fractional Hamilton equations of motion. Using equations (9), (11) and (14), it can be demonstrated that equations (15) lead to equation (8) (i.e. the correct Euler–Lagrange equation of motion).

Case 2. Change of complete set of GVDs. The Hamiltonian formulation, where the complete set of GVDs is replaced with the complete set of FCMDs, can be obtained from the above formulation by replacing both n_l and n_r with n. Thus, for this case, using equation (11) the Hamiltonian density function is given as

$$\mathcal{H} = \sum_{k=1}^{n} \left(\pi_{\alpha_k}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right) + \pi_{\beta_k}\left({}_{-}^{C}\partial_k^{\beta}\phi\right)\right) - \mathcal{L} \tag{16}$$

and using equation (15), the Hamiltonian equations of motion are given as

$$\begin{aligned} &\frac{\partial\mathcal{H}}{\partial\pi_{\alpha_k}} = {}_{+}^{C}\partial_k^{\alpha}\phi, \qquad \frac{\partial\mathcal{H}}{\partial\pi_{\beta_k}} = {}_{-}^{C}\partial_k^{\beta}\phi, \qquad \frac{\partial\mathcal{H}}{\partial x_k} = -\frac{\partial\mathcal{L}}{\partial x_k}, \\ &\frac{\partial\mathcal{H}}{\partial\phi} = \sum_{k=1}^{n}\left({}_{-}\partial_k^{\alpha}\pi_{\alpha_k} + {}_{+}\partial_k^{\beta}\pi_{\beta_k}\right), \qquad k = 1,\ldots,n. \end{aligned} \tag{17}$$

Case 3. Change of GVDs when the Lagrangian density function is a linear function of the GVDs. The forgoing approach cannot be used to obtain Hamiltonian density function if the Lagrangian density function is a linear function of the GVDs. To demonstrate this, consider a Lagrangian density function $\mathcal{L}(\phi, {}_{+}^{C}\partial_k^{\alpha}\phi, \bar{\phi})$ as

$$\mathcal{L}\left(\phi, {}_{+}^{C}\partial_k^{\alpha}\phi, {}_{-}^{C}\partial_k^{\beta}\phi, \bar{\phi}\right) = \bar{\phi}\sum_{k=1}^{n}\left(a_k\left({}_{+}^{C}\partial_k^{\alpha}\phi\right) + b_k\left({}_{-}^{C}\partial_k^{\beta}\phi\right)\right) + c\bar{\phi}\phi, \tag{18}$$

where c, a_k, b_k ($k = 1,\ldots,n$), are some constants and $\bar{\phi} \equiv \bar{\phi}(x_k)$ is some function of x_k ($k = 1,\ldots,n$). As we shall see, such Lagrangian density functions arise in our Dirac field

formulation presented in the next section. For this case, the first Euler–Lagrange equation is given as

$$\frac{\partial \mathcal{L}}{\partial \bar{\phi}} = \sum_{k=1}^{n} \left(a_k \left({}_{+}^{C}\partial_k^{\alpha} \phi \right) + b_k \left({}_{-}^{C}\partial_k^{\beta} \phi \right) \right) + c\phi = 0, \tag{19}$$

the FCMDs are given as

$$\pi_{\alpha_k} = \frac{\partial \mathcal{L}}{\partial {}_{+}^{C}\partial_k^{\alpha} \phi} = \bar{\phi} a_k, \qquad \pi_{\beta_k} = \frac{\partial \mathcal{L}}{\partial {}_{-}^{C}\partial_k^{\beta} \phi} = \bar{\phi} b_k, \qquad k = 1, \ldots, n \tag{20}$$

and the Hamiltonian density function is given as

$$\mathcal{H} = \sum_{k=1}^{n} \left(\pi_{\alpha_k} \left({}_{+}^{C}\partial_k^{\alpha} \phi \right) + \pi_{\beta_k} \left({}_{-}^{C}\partial_k^{\beta} \phi \right) \right) - \mathcal{L} = -c\bar{\phi}\phi. \tag{21}$$

It could be verified that this Hamiltonian does not lead to the correct Euler–Lagrange equation of motion.

To overcome this problem, we treat the definition of the FCMDs as constraints and add these constraints to the Hamiltonian density function using Lagrange multiplier to obtain the new Hamiltonian density function. It can be demonstrated that a new Hamiltonian density function gives the correct equation of motion. To accomplish this, consider the Lagrangian density function $\mathcal{L}$ defined by equation (18) and assume that we want to replace the GVD ${}_{+}^{C}\partial_1^{\alpha} \phi$ with the corresponding FCMD. In this case, the FCMD corresponding to the GVD ${}_{+}^{C}\partial_1^{\alpha} \phi$ is given as

$$\pi_{\alpha_1} = \frac{\partial \mathcal{L}}{\partial \left({}_{+}^{C}\partial_1^{\alpha} \phi \right)} = \bar{\phi} a_1, \tag{22}$$

which leads to the constraints as

$$\pi_{\alpha_1} - \bar{\phi} a_1 = 0 \tag{23}$$

and the Hamiltonian as

$$\mathcal{H} = -\bar{\phi} \sum_{k=2}^{n} a_k \left({}_{+}^{C}\partial_k^{\alpha} \phi \right) - \bar{\phi} \sum_{k=1}^{n} b_k \left({}_{-}^{C}\partial_k^{\beta} \phi \right) - c\bar{\phi}\phi + \lambda(\pi_{\alpha_1} - \bar{\phi} a_1), \tag{24}$$

where λ is a Lagrange multiplier.

Here only one GVD has been replaced with one FCMD and therefore, we must consider the formulation presented in case 1 above. Thus, using equations (15) and (24), we obtain

$$\begin{aligned} &\frac{\partial \mathcal{H}}{\partial \bar{\phi}} = -\sum_{k=2}^{n} a_k \left({}_{+}^{C}\partial_k^{\alpha} \phi \right) - \sum_{k=1}^{n} b_k \left({}_{-}^{C}\partial_k^{\beta} \phi \right) - c\phi - \lambda a_1 = 0, \\ &{}_{+}^{C}\partial_1^{\alpha} \phi = \frac{\partial \mathcal{H}}{\partial \pi_{\alpha_1}} = \lambda, \end{aligned} \tag{25}$$

which immediately lead to equation (19), i.e. the correct equation of motion. Following the same approach, it can be demonstrated that the approach works when two or more GVDs are replaced with the corresponding FCMD.

In the next section, the formulations of this and the previous section will be used to develop a Lagrangian and a Hamiltonian of a fractional Dirac field and the fractional Dirac equations.

4. Dirac's field with Caputo fractional derivatives

We will now apply the Lagrangian and the Hamiltonian formulations developed in sections 2 and 3 to derive fractional Dirac equation for a relativistic particle. We shall limit our discussion to four-dimensional system (the first three for space, x_1, x_2 and x_3, and the fourth for time, $x_4 = it$, where i is the imaginary unit. Note that we are considering the units that take the speed of light equal to 1). The Greek indices μ, λ, ν, etc will range from 1 to 4, the Roman indices i, j, k, etc will range from 1 to 3 and, unless specifically stated, the repeated indices will represent summation. Following this convention, we propose the following Lagrangian density field:

$$\mathcal{L} = m^{\alpha}\bar{\Psi}\Psi + a\bar{\Psi}\gamma^{\mu}\left({}^{C}_{+}\partial^{\alpha}_{\mu}\Psi\right) + b\left({}_{-}\partial^{\alpha}_{\mu}\bar{\Psi}\right)\gamma^{\mu}\Psi, \tag{26}$$

where Ψ is a wavefunction (the bold character Ψ suggests that it could be a vector function; in the literature, it is also called a spinor), m is the mass of the particle, α $(= \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4)$ is the order of the fractional derivative, γ^{μ} are matrices, $\bar{\Psi}$ is the adjoint wavefunction, and a and b are coefficients such that $a + b = 1$. The dimensions of Ψ, $\bar{\Psi}$ and γ^{μ} depend on the order of the derivatives and the way the algebra of matrices γ^{μ} is developed. For example, if the algebra of γ^{μ} is developed by taking the square root of the Klein–Gordon equation, then the minimum dimensions of γ^{μ} turn out to be 4 by 4. This also applies to fractional Dirac equations when they are obtained by taking the square root of a fractional Klein–Gordon equation. On the other hand, as reported by Raspini, if the algebra of γ^{μ} is developed by taking cube roots of the Klein–Gordon equation, the minimum dimensions of matrices γ^{μ} turn out to be 9 by 9. For the sake of generality, we shall keep the two terms Ψ and γ^{μ} general and consider special cases as necessary.

Using equations (8) and (26), the Euler–Lagrange equations for variables $\bar{\Psi}$ and Ψ are given as

$$\gamma^{\mu}\left({}^{C}_{+}\partial^{\alpha}_{\mu}\Psi\right) + m^{\alpha}\Psi = 0 \tag{27}$$

and

$$\left({}_{-}\partial^{\alpha}_{\mu}\bar{\Psi}\right)\gamma^{\mu} + m^{\alpha}\bar{\Psi} = 0. \tag{28}$$

Equation (27) is the desired fractional Dirac equation and equation (28) is its adjoint form. For $\alpha = 1$, we obtain the standard Dirac equation, and for $\alpha = 2/3$ and assuming that γ^{μ} are the same as those developed in [28], we obtain the fractional Dirac equation of order 2/3 developed by Raspini. It will be shown in the next section that the fractional Klein–Gordon equation gives the same fractional Dirac equation as that given in equation (27). Note that a and b do not appear in equations (27) and (28). This suggests that we can take different values of a and b to generate a family of Lagrangian density functions each of which would give the same fractional Dirac equation and its adjoint as long as the condition $a + b = 1$ is satisfied.

In the discussion to follow, we shall take $a = 1$ and $b = 0$. Let us assume that we are interested in time evolution only, and we want to develop the corresponding Hamiltonian. For this case, the corresponding canonical momenta are given as

$$\pi_4 = \frac{\partial\mathcal{L}}{\partial\left({}^{C}_{+}\partial^{\alpha}_{4}\Psi\right)} = \bar{\Psi}\gamma^{4}, \qquad \bar{\pi}_4 = \frac{\partial\mathcal{L}}{\partial\left({}^{C}_{+}\partial^{\alpha}_{4}\bar{\Psi}\right)} = 0. \tag{29}$$

Using the Lagrange multiplier technique discussed in section 3, the fractional Hamiltonian density function for this system is given by

$$\mathcal{H}_f = -\bar{\Psi}\left(\gamma^{k}\left({}^{C}_{+}\partial^{\alpha}_{k}\Psi\right) + m^{\alpha}\Psi\right) + \left(\pi_4 - \bar{\Psi}\gamma^{4}\right)\lambda + \bar{\lambda}\bar{\pi}_4, \tag{30}$$

where λ and $\bar{\lambda}$ are the Lagrange multipliers.

Using equations (15) and (30), the fractional Hamilton equations are given as

$$\frac{\partial \mathcal{H}_f}{\partial \bar{\Psi}} = -\left(\gamma^k \left({}_+^C\partial_k^\alpha \Psi\right) + m^\alpha \Psi\right) - \gamma^4 \lambda = 0, \qquad \frac{\partial \mathcal{H}_f}{\partial \pi_4} = \lambda = {}_+^C\partial_4^\alpha \Psi \tag{31}$$

and

$$\frac{\partial \mathcal{H}_f}{\partial \Psi} = -\bar{\Psi} m^\alpha = {}_-\partial_4^\alpha \pi_4 + {}_-\partial_k^\alpha \bar{\Psi} \gamma_k. \tag{32}$$

It could be verified that equations (31) and (32) led to equations (27) and (28). Thus, the proposed Hamiltonian density function leads to the correct set of differential equations.

In the next section, we derive the FDE from the fractional Klein–Gordon, find the eigensolutions of the FDE and discuss the resulting consequences.

5. Fractional Klein–Gordon and Dirac equations, and the solutions of the FDE

In this section we propose a fractional Klein–Gordon equation (FKGE) as

$$\left[\left({}_+^C\partial_\mu^\alpha\right)\left({}_+^C\partial_\mu^\alpha\right) - m^{2\alpha}\right]\Psi(\mathbf{x}, t) = 0. \tag{33}$$

Here we use Einstein's summation rule for μ only. Note that here we have defined the fractional derivative in the Caputo sense. This allows us to find solutions of some fractional differential equations in closed form. Further, for $\alpha = 1$, equation (33) reduces to the standard Klein–Gordon equation.

The fractional momenta operator p_α^μ is defined as

$$p_\alpha^\mu = -\mathrm{i}\left({}_+^C\partial_\mu^\alpha\right). \tag{34}$$

Hence, (33) can be written as

$$\left[p_\alpha^\mu p_\alpha^\mu + m^{2\alpha}\right]\Psi(\mathbf{x}, t) = 0. \tag{35}$$

The fractional Klein–Gordon equation defined by equation (35) is quadratic in the momenta operator p_α^μ. Let us assume that

$$\left[\mathrm{i}\gamma^\mu p_\alpha^\mu + m^\alpha\right]\Psi(\mathbf{x}, t) = 0 \tag{36}$$

is a solution to equation (35). This equation is called the fractional Dirac equation (FDE) of order α. To find the algebra of γ^μ and the solution of equation (36), let us apply the operator $\left[\mathrm{i}\gamma^\mu p_\alpha^\mu - m^\alpha\right]$ to it from left to obtain

$$\left[-\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\alpha^\mu p_\alpha^\nu - m^{2\alpha}\right]\Psi(\mathbf{x}, t) = 0. \tag{37}$$

In order to recover $\left[p_\alpha^\mu p_\alpha^\mu + m^{2\alpha}\right]\Psi(\mathbf{x}, t) = 0$, we require the following algebra for γ's

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta_{\mu\nu}, \tag{38}$$

which is the same as that required of γ's in the ordinary Dirac equations. Here $\delta_{\mu\nu}$ is the Kronecker-delta function. Thus, the γ's appearing here and those appearing in the ordinary Dirac equations are the same. The explicit representations of the γ^μ are

$$\gamma^4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{39}$$

and

$$\gamma^k = \begin{pmatrix} 0 & -\mathrm{i}\sigma_k \\ \mathrm{i}\sigma_k & 0 \end{pmatrix}, \tag{40}$$

where I is a 2×2 identity matrix, σ_k are the 2×2 *Pauli matrices* defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{41}$$

and the wavefunction Ψ is given as

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{42}$$

Using equations (34) and (36), we obtain

$$\left[\gamma^{\mu}\left({}^{C}_{+}\partial^{\alpha}_{\mu}\right) + m^{\alpha}\right]\Psi(\mathbf{x}, t) = 0, \tag{43}$$

which is exactly the same as that given in equation (27). Thus, the approach of taking the root of Klein–Gordon discussed here and the variational principle approach discussed in section 4 give the same fractional Dirac equations.

Multiply γ^4 to equation (43) from left, separating the time and the spatial parts, and using the fact that $(\gamma^4)^2 = 1$, we obtain

$$-\left({}^{C}_{+}\partial^{\alpha}_{4}\Psi\right) = \tau^{k}\left({}^{C}_{+}\partial^{\alpha}_{k}\Psi\right) + \gamma^{4}m^{\alpha}\Psi, \tag{44}$$

where $\tau^k = \gamma^4\gamma^k$. Assuming that the method of separation of variables can be applied, we take $\Psi(\mathbf{x}, t) = \varphi(t)\Phi(\mathbf{x})$. Separating the time and the space parts and assuming that each is equal to E, we obtain

$$\left({}^{C}_{+}\partial^{\alpha}_{4}\right)\varphi(t) = -E\varphi(t) \tag{45}$$

and

$$\left(\mathrm{i}\tau^{k}p^{k}_{\alpha} + \gamma^{4}m^{\alpha}\right)\Phi(\mathbf{x}) = E\Phi(\mathbf{x}). \tag{46}$$

Here we have replaced ${}^{C}_{+}\partial^{\alpha}_{k}$ with $\mathrm{i}p^{k}_{\alpha}$ (see equation (34)).

To find the solution of equation (45), we assume that $\varphi(t = 0) = 1$, and then the time-dependent solution $\varphi(t)$ is given by [3]

$$\varphi(t) = E_{\alpha}(-E(it)^{\alpha}), \tag{47}$$

where E_α is the Mittag–Leffler function defined as

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{(z)^{n}}{\Gamma(\alpha n + 1)}. \tag{48}$$

Also, from equation (34), we have

$$-\mathrm{i}\left({}^{C}_{+}\partial^{\alpha}_{k}\right)\Phi(\mathbf{x}) = p^{k}_{\alpha}\Phi(\mathbf{x}). \tag{49}$$

Using the separation of variables method once again, and assuming that $\Phi(\mathbf{x}) = X_1(x_1)X_2(x_2)X_3(x_3)U$, we obtain the fractional differential equations for $X_k(x_k)$ $(k = 1, 2, 3)$ as

$$-\mathrm{i}{}^{C}_{+}\partial^{\alpha}_{k}X_k(x_k) = p^{k}_{\alpha}X_k(x_k) \quad (k = 1, 2, 3). \tag{50}$$

Here, no sum is taken over k. Let us assume that $X_k(0) = 1$ $(k = 1, 2, 3)$. For this case, the solution of (50) is given as $X_k(x_k) = E_{\alpha}\left(\mathrm{i}p^{k}_{\alpha}(x_k)^{\alpha}\right)$, $k = 1, 2, 3$, and accordingly, the solution for $\Phi(\mathbf{x})$ is given as

$$\Phi(\mathbf{x}) = E_{\alpha}\left(\mathrm{i}p^{1}_{\alpha}(x_1)^{\alpha}\right)E_{\alpha}\left(\mathrm{i}p^{2}_{\alpha}(x_2)^{\alpha}\right)E_{\alpha}\left(\mathrm{i}p^{3}_{\alpha}(x_3)^{\alpha}\right)U, \tag{51}$$

and the spacetime spinor wavefunction for fractional Dirac equation is given by

$$\Psi_{fD} = E_{\alpha}(-E(\mathrm{i}t)^{\alpha})E_{\alpha}\left(\mathrm{i}p^{1}_{\alpha}(x_1)^{\alpha}\right)E_{\alpha}\left(\mathrm{i}p^{2}_{\alpha}(x_2)^{\alpha}\right)E_{\alpha}\left(\mathrm{i}p^{3}_{\alpha}(x_3)^{\alpha}\right)U, \tag{52}$$

where U is the spinor wave vector defined as

$$U = \begin{pmatrix} \chi \\ \eta \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}. \tag{53}$$

Now, equation (46) can be written as

$$\begin{pmatrix} (m^\alpha - E)I & \sigma^k p_\alpha^k \\ -\sigma^j p_\alpha^j & (m^\alpha + E)I \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} = 0. \tag{54}$$

For nontrivial solution of equation (54), the determinant of the coefficient matrix must vanish. This leads to

$$(m^\alpha - E)(m^\alpha + E) + \left(\sigma^k p_\alpha^k\right)\left(\sigma^j p_\alpha^j\right) = 0 \tag{55}$$

from which we obtain the eigenvalues $E_\pm^{fD}$ of the fractional Dirac equation as

$$E_\pm^{fD} = \pm\sqrt{p_\alpha^2 + m^{2\alpha}}. \tag{56}$$

For $\alpha = 1$, we have the energy eigenvalues for the regular Dirac equation as

$$E_\pm^{D} = \pm\sqrt{p^2 + m^2}. \tag{57}$$

Equation (54) can be solved using standard techniques to determine four eigenvectors U corresponding to the eigenvalues $E_\pm^{fD}$. These vectors are

$$\begin{aligned} U_\uparrow^+ &= N\left[1 \quad 0 \quad \tfrac{p_\alpha^3}{E_+^{fD}+m^\alpha} \quad \tfrac{p_\alpha^+}{E_+^{fD}+m^\alpha}\right]^T, \\ U_\downarrow^+ &= N\left[0 \quad 1 \quad \tfrac{p_\alpha^-}{E_+^{fD}+m^\alpha} \quad \tfrac{-p_\alpha^3}{E_+^{fD}+m^\alpha}\right]^T, \\ U_\uparrow^- &= N\left[\tfrac{p_\alpha^3}{E_-^{fD}-m^\alpha} \quad \tfrac{p_\alpha^+}{E_-^{fD}-m^\alpha} \quad 1 \quad 0\right]^T, \\ U_\downarrow^- &= N\left[\tfrac{p_\alpha^+}{E_-^{fD}-m^\alpha} \quad \tfrac{-p_\alpha^3}{E_-^{fD}-m^\alpha} \quad 1 \quad 0\right]^T, \end{aligned} \tag{58}$$

where $p_\alpha^\pm = p_\alpha^1 \pm \mathrm{i}p_\alpha^2$ and N is the normalization coefficient such that $U^{*T}U = 1$. Here the superscript $*$ represents the complex conjugate of the vector. It can be shown that

$$N = \left[1 + \frac{p_\alpha^k p_\alpha^k}{(|E^{fD}| + m^\alpha)^2}\right]^{1/2} \tag{59}$$

and that $U_\uparrow^+$, $U_\downarrow^+$, $U_\uparrow^-$ and $U_\downarrow^-$ are orthonormal vectors, and they can be used to find the complete solution.

6. Path integral quantization of the fractional Dirac equation

The path integral method is an alternative formulation of quantum mechanics and leads us to the same results obtained by canonical quantization (i.e. the operator formalism of quantum mechanics). However, for some systems, the canonical method is quite awkward to formulate and to use. This holds true for quantization of fields with constraints, and the problem of non-Abelian gauge field theories. The operators are avoided by the use of infinite product of integrals (path integral), which nowadays is the preferred route to field quantization.

For classical fields ϕ, the path integral is given by the transition amplitude [30, 31]

$$K = \int D\phi\, \mathrm{e}^{\mathrm{i}S/\hbar}. \tag{60}$$

The starting point of the quantization process, in the path integral quantization for field systems, is to calculate the integrable action function integral S [32] which gives the genuine equations of motion. For fractional field systems, the action function is given by

$$S(\phi) = \int \mathcal{L}\left(\phi(x_k), \left({}_{+}^{C}\partial_k^{\alpha}\right)\phi(x_k), \left({}_{-}^{C}\partial_k^{\beta}\right)\phi(x_k), x_k\right)(\mathrm{d}x_k). \tag{61}$$

For the fractional Dirac field, we calculate the functional action as

$$S = \int \left\{\bar{\Psi}\left(\gamma^{\mu}\left({}_{+}^{C}\partial_{\mu}^{\alpha}\Psi\right) + m^{\alpha}\Psi\right)\right\}\mathrm{d}^4x. \tag{62}$$

The above formulation leads us to obtain the path integral quantization for the fractional Dirac field as

$$K = \int \mathrm{d}\bar{\Psi}\,\mathrm{d}\Psi \exp\left[\mathrm{i}/\hbar \int \left\{\bar{\Psi}\left(\gamma^{\mu}\left({}_{+}^{C}\partial_{\mu}^{\alpha}\Psi\right) + m^{\alpha}\Psi\right)\right\}\mathrm{d}^4x\right]. \tag{63}$$

The path integral representation (63) is an integration over the canonical field variables $\bar{\Psi}$ and Ψ. One should notice that as $\alpha \to 1$, we have the path integral quantization for the regular free Dirac field.

This path integral scheme can be applied for any fractional mechanical system, in the same sense as that of applying the path integral for Lagrangian systems with integer order derivatives and this topic will be discussed in a later paper.

7. Conclusions

Fractional derivatives and integrals have the specific property that they contain in the limit $\alpha \to 1$ the classical ones. As a result the fractional models contain as a limit the classical ones. This property makes the fractional calculus an important candidate to describe the anomalous processes. Having this in mind, in this paper, we have presented a fractional Dirac equation. We derived the fractional Dirac equation using two methods, first using a variational principle and then by taking the square root of a fractional Klein–Gordon equation. We demonstrated that both methods give the same fractional Dirac equation. In the process, we extended the derivation of the usual Euler–Lagrange and Hamilton equations of motion for the classical field to fields defined using Caputo fractional derivatives. We also presented the eigensolution of the fractional Dirac equation. We have also presented the path integral quantization for the fractional Dirac field with a hope that it will allow further development of fractional relativistic quantum mechanics.

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